G_2–instantons on generalised Kummer constructions. I

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January 26, 2012

Abstract

In this article we introduce a method to construct G_2–instantons on G_2-manifolds arising from Joyce's generalised Kummer construction [Joy96]. The method is based on gluing anti-self-dual instantons on ALE spaces to flat bundles on G_2-orbifolds of the form T^7/Γ. We use this construction to produce non-trivial examples of G_2–instantons. Finally, we discuss the relevance of our work to the computation of a conjectural G_2 Casson invariant as well as potential future applications.

1 Introduction

The seminal paper [DT98] of Donaldson-Thomas has inspired a considerable amount of work related to gauge theory in higher dimensions. Tian [Tia00] and Tao–Tian [TT04] made significant progress on important foundational analytical questions. Recent work of Donaldson–Segal [DS09] and Haydys [Hay11b, Hay11a] shed some light on the shape of the theories to be expected.

In this article we will focus on the study of gauge theory on G_2-manifolds. These are 7-manifolds equipped with a torsion-free G_2-structure. The G_2-structure allows to define a special class of connections, called G_2–instantons. These share many formal properties with flat connections on 3–manifolds and it is expected that there are G_2–analogues of those 3–manifold invariants related to "counting flat connections", i.e., the Casson invariant, instanton Floer homology, etc.

So far non-trivial examples of G_2–instantons are rather rare. By exploiting the special geometry of the known G_2–manifolds some progress has been made recently. At the time of writing, there are essentially two methods for constructing compact G_2–manifolds in the literature. Both yield G_2–manifolds close to degenerate limits. One is Kovalev's twisted connected sum construction [Kov03], which produces G_2–manifolds with "long necks" from certain pairs of Calabi-Yau 3-folds with asymptotically cylindrical ends. A technique for constructing G_2–instantons on Kovalev's G_2–manifolds has recently been introduced by Sá Earp [SE11a, SE11b]. The other (and historically the first) method for constructing G_2–manifolds is due to Joyce and is based on desingularising G_2–orbifolds [Joy96]. In this article we introduce a method to construct G_2–instantons on Joyce's manifolds.

To setup the framework for our construction, let us briefly review the geometry of Joyce's construction. Give T^7 the structure of a flat G_2–orbifold and let Γ be a finite group of diffeomorphisms of T^7 preserving the flat G_2–structure. Then $Y_0 = T^7/\Gamma$ is a flat G_2–orbifold. In general, the singular set S of Y_0 can be quite complicated. For the purpose of this article, we assume that each of the connected components S_j of S has a neighbourhood T_j modelled on $(T^3 \times \mathbf{C}^2/G_j)/H_j$. Here G_j is a non-trivial finite subgroup of $\mathrm{SU}(2)$ and H_j is a finite subgroup of isometries of $T^3 \times \mathbf{C}^2/G_j$ acting freely on T^3. We call G_2–orbifolds Y_0 satisfying this condition

admissible. Now, for each j, pick an ALE space X_j asymptotic to $\mathbf{C}^2/G_j$ and an isometric action ρ_j of H_j on X_j which is asymptotic to the action of H_j on $\mathbf{C}^2/G_j$. Given such a collection $\{(X_j, \rho_j)\}$ of *resolution data* for Y_0 and a small parameter $t > 0$, one can rather explicitly construct a compact 7–manifold Y_t together with a G_2–structure that is close to being torsion-free. Joyce proved that this G_2–structure can be perturbed slightly to yield a torsion-free G_2–structure provided $t > 0$ is sufficiently small.

We will introduce a notion of *gluing data* $((E_0, \theta), \{(x_j, f_j)\}, \{(E_j, A_j, \tilde{\rho}_j, m_j)\})$ compatible with resolution data $\{(X_j, \rho_j)\}$ for Y_0, consisting of a flat connection θ on a G–bundle E_0 over Y_0 and, for each j, a G–bundle E_j together with an ASD instanton A_j framed at infinity as well as various additional data satisfying a number of compatibility conditions. With this piece of notation in place, the main result of this article is the following.

Theorem 1.1. *Let Y_0 be an admissible flat G_2–orbifold, let $\{(X_j, \rho_j)\}$ be resolution data for Y_0 and let $((E_0, \theta), \{(x_j, f_j)\}, \{(E_j, A_j, \tilde{\rho}_j, m_j)\})$ be compatible gluing data. Then there is a constant $\kappa > 0$ such that for $t \in (0, \kappa)$ one can construct a G–bundle E_t on Y_t together with a connection A_t satisfying*

$$p_1(E_t) = -\sum_j k_j \,\mathrm{PD}[S_j] \quad \textit{with} \quad k_j = \frac{1}{8\pi^2} \int_{X_j} |F_{A_j}|^2$$

and

$$\langle w_2(E_t), \Sigma \rangle = \langle w_2(E_j), \Sigma \rangle$$

for each $\Sigma \in H_2(X_j)^{H_j} \subset H_2(Y_t)$. Here $[S_j] \in H_3(Y_t)$ is the homology class induced by the component S_j of the singular set S.

If θ is regular as a G_2–instanton and each of the A_j is infinitesimally rigid, then for $t \in (0, \kappa)$ there is a small perturbation $a_t \in \Omega^1(Y_t, \mathfrak{g}_{E_t})$ such that $A_t + a_t$ is a regular G_2–instanton.

The analysis involved in the proof of Theorem 1.1 is similar to unpublished work of Brendle on the Yang–Mills equation in higher dimension [Bre03] and Pacard–Ritoré's work on the Allen–Cahn equation [PR03], however our assumptions lead to some simplifications. From a geometric perspective our result can be viewed as a higher dimensional analogue Kronheimer's work on ASD instantons on Kummer surfaces [Kro91].

Here is an outline of this article. To benefit the reader Sections 2, 3, 4 and 5 contain some foundational material on G_2–manifolds and G_2–instantons as well as brief reviews of Joyce's generalised Kummer construction and Kronheimer's and Nakajima's work on ASD instantons on ALE spaces. In Section 6 we define the notion of gluing data and prove the first part of Theorem 1.1. We also introduce weighted Hölder spaces adapted to the problem at hand and prove that the connection A_t is a good approximation to a G_2–instanton. In Section 7 we set up the analytical problem underlying the proof of Theorem 1.1 and discuss the properties of a model for the linearised problem. We complete the proof of Theorem 1.1 in Section 8. A number of examples with $G = \mathrm{SO}(3)$ are constructed in Section 9, before we close our exposition with a short discussion of the relevance of our work to computing a conjectural G_2 Casson invariant and hint at potential future applications in Section 10.

Acknowledgements This article is the outcome of work undertaken by the author for his PhD thesis at Imperial College London, funded by the European Commission. I would like to thank my supervisor Simon Donaldson for guidance and inspiration.

2 Review of G_2–manifolds

The Lie group G_2 can be defined as the subgroup of elements of GL(7) fixing the 3–form

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}.$$

Here $\mathrm{d}x^{ijk}$ is a shorthand for $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ and $x_1, \ldots, x_7$ are standard coordinates on $\mathbf{R}^7$. The particular choice of ϕ_0 is not important. In fact, any non-degenerate 3–form ϕ is equivalent to ϕ_0 under a change of coordinates, see e.g. [SW10, Theorem 3.2]. Here we say that ϕ is non-degenerate if for each non-zero vector $u \in \mathbf{R}^7$ the 2–form $i(u)\phi$ on $\mathbf{R}^7/\langle u\rangle$ is symplectic. G_2 naturally is a subgroup of SO(7). To see this, note that any element of GL(7) that preserves ϕ_0 also preserves the standard inner product and the standard orientation of $\mathbf{R}^7$. This follows from the identity

$$i(u)\phi_0 \wedge i(v)\phi_0 \wedge \phi_0 = 6 g_0(u,v)\mathrm{vol}_0.$$

In particular, every non-degenerate 3–form ϕ on a 7–dimensional vector space naturally induces an inner product and an orientation. As an aside we should point out here that non-degenerate 3–forms form one of two open orbits of GL(7) in $\Lambda^3(\mathbf{R}^7)^*$. For ϕ in the other open orbit, the above formula yields an indefinite metric of signature $(3,4)$. In particular, if we take $u = v$ to be a light-like vector, then $i(u)\phi$ is not a symplectic form on $\mathbf{R}^7/\langle u\rangle$.

A G_2*–manifold* is 7–manifold equipped with a torsion-free G_2–structure, i.e., a reduction of the structure group of TY from GL(7) to G_2 with vanishing intrinsic torsion. From the above it is clear that a G_2–structure on Y is equivalent to a 3–form ϕ on Y, which is non-degenerate at every point of Y. The condition to be torsion-free means that ϕ is parallel with respect to the Levi–Civita connection associated with the metric induced by ϕ. This can also be understood as follows. Let $\mathscr{P} \subset \Omega^3(Y)$ denote the subspace of all non-degenerate 3–forms ϕ. Then there is a *non-linear* map $\Theta \colon \mathscr{P} \to \Omega^4(Y)$ defined by

$$\Theta(\phi) = *_\phi \phi.$$

Here $*_\phi$ is the Hodge-$*$-operator coming from the inner product and orientation induced by ϕ. Now the G_2–structure induced by ϕ is torsion-free if and only if

$$\mathrm{d}\phi = 0, \quad \mathrm{d}\Theta(\phi) = 0.$$

The proof of this fact is an exercise in linear algebra.

Since G_2–manifolds have holonomy contained in G_2 which can also be viewed as the subgroup of Spin(7) fixing a non-trivial spinor, G_2-manifolds are spin and admit a non-trivial parallel spinor. In particular, they are Ricci-flat. Moreover, G_2–manifold naturally carry a pair of calibrations: the *associative calibration* ϕ and the *coassociative calibration* $\psi := \Theta(\phi)$. This should be enough to convince the reader, that G_2–manifolds are interesting geometric objects worthy of further study.

Simple examples of G_2–manifolds are T^7 with the constant torsion-free G_2–structure ϕ_0 and products of the form $T^3 \times \Sigma$, where Σ is a hyperkähler surface, with the torsion-free G_2–structure induced by

$$\phi_0 = \delta_1 \wedge \delta_2 \wedge \delta_3 + \delta_1 \wedge \omega_1 + \delta_2 \wedge \omega_2 - \delta_3 \wedge \omega_3$$

where $\delta_1, \delta_2, \delta_3$ are constant orthonormal 1–forms on T^3 and $\omega_1, \omega_2, \omega_3$ define the hyperkähler structure on Σ. These examples all have holonomy strictly contained in G_2. Compact examples with full holonomy G_2 are much harder to come by. In Section 4, we will review Joyce's generalised Kummer construction which produced the first examples of G_2–manifolds of this kind.

3 Gauge theory on G_2–manifolds

Let (Y,ϕ) be a G_2–manifold and let E be a G–bundle over Y. Denote by $\psi := \Theta(\phi)$ the corresponding coassociative calibration. A connection $A \in \mathscr{A}(E)$ on E is called a G_2*–instanton* if it satisfies the equation

$$F_A \wedge \psi = 0.$$

This equation is equivalent to $*(F_A \wedge \phi) = -F_A$ and thus G_2–instantons are solutions of the Yang–Mills equation. In fact, if Y is compact, then G_2-instantons are absolute minima of the Yang–Mills functional.

From an analytical point of view the above equations are not satisfying because they are not manifestly elliptic. This issue is resolved by considering instead the equation

$$*(F_A \wedge \psi) + \mathrm{d}_A \xi = 0$$

where $\xi \in \Omega^0(Y, \mathfrak{g}_E)$. If Y is compact (or under certain decay assumptions) it follows from the Bianchi identity and $\mathrm{d}\psi = 0$ that $\mathrm{d}_A \xi = 0$. Thus this it is equivalent to the G_2–instanton equation. Once gauge invariance is accounted for the equation becomes elliptic. The gauge fixed linearisation at a G_2–instanton A is given by the operator $L_A \colon \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E) \to \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E)$ defined by

$$L_A = \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix}.$$

It is easy to see that L_A is a self-adjoint elliptic operator. In particular, the virtual dimension of the moduli space

$$\mathscr{M}(E,\phi) = \{A \in \mathscr{A}(E) : F_A \wedge \psi = 0\} / \mathscr{G}$$

is zero. It is very tempting to try to define a G_2 *Casson invariant* by "counting" $\mathscr{M} = \mathscr{M}(E,\phi)$. If every G_2–instanton A on E is *regular*, this means that L_A is an isomorphism, then $\mathscr{M}$ is a smooth zero-dimensional manifold, i.e., a discrete set. If $\mathscr{M}$ was compact, then up to questions of orientation we could indeed count $\mathscr{M}$.

Whether there is a rigorous general definition of a G_2 Casson invariant and whether it is, in fact, invariant under isotopies the G_2–structure is an open question. A brief discussion of parts of this circle of ideas can be found in Donaldson–Segal [DS09, Section 6].

Simple examples of G_2–instantons are flat connections and ASD instantons on K3 surfaces Σ pulled back to $T^3 \times \Sigma$. Moreover, the Levi–Civita connection on a G_2–manifolds is a G_2–instanton. We will construct non-trivial examples using Theorem 1.1 in Section 9.

4 Joyce's generalised Kummer construction

Consider T^7 together with a constant 3–form ϕ_0 defining a G_2–structure and let Γ be a finite group of diffeomorphisms of T^7 preserving ϕ_0. Then $Y_0 = T^7/\Gamma$ naturally is a flat G_2–orbifold. Denote by S the singular set of Y_0 and denote by $S_1, \ldots, S_k$ its connected components. In general, S can be rather complicated. In this article we make the assumption that each S_j has a neighbourhood isometric to a neighbourhood of the singular set of $(T^3 \times \mathbf{C}^2/G_j)/H_j$. Here G_j is a non-trivial finite subgroup of $\mathrm{SU}(2)$ and H_j is a finite subgroup of isometries of $T^3 \times \mathbf{C}^2/G_j$ acting freely on T^3. G_2–orbifolds Y_0 if this kind will be called *admissible*.

Let Y_0 be an admissible flat G_2–orbifold. Then there is a constant $\zeta > 0$ such that if we denote by T the set of points at distance less that ζ to S, then T decomposes into connected components $T_1, \dots, T_k$, such that T_j contains S_j and is isometric to $(T^3 \times B^4_\zeta/G_j)/H_j$. On T_j we can write

$$\phi_0 = \delta_1 \wedge \delta_2 \wedge \delta_3 + \delta_1 \wedge \omega_1 + \delta_2 \wedge \omega_2 - \delta_3 \wedge \omega_3$$

where $\delta_1, \delta_2, \delta_3$ are constant orthonormal 1–forms on T^3 and $\omega_1, \omega_2, \omega_3$ can be written as $\omega_1 = e^1 \wedge e^2 + e^3 \wedge e^4$, $\omega_2 = e^1 \wedge e^3 - e^2 \wedge e^4$, $\omega_3 = e^1 \wedge e^4 + e^2 \wedge e^3$ in an oriented orthonormal basis $(e^1, \dots, e^4)$ for $(\mathbf{C}^2)^*$.

We will remove the singularity along S_j by replacing $\mathbf{C}^2/G_j$ with an *ALE space asymptotic to* $\mathbf{C}^2/G_j$. That is a hyperkähler manifold $(X, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ together with a continuous map $\pi\colon X \to \mathbf{C}^2/G_j$ inducing a diffeomorphisms from $X \setminus \pi^{-1}(0)$ to $(\mathbf{C}^2 \setminus \{0\})/G_j$ such that

$$\nabla^k(\pi_* \tilde{\omega}_i - \omega_i) = O(r^{-4-k}),$$

as $r \to \infty$ for $i = 1, 2, 3$ and $k \geq 0$. Here $r\colon \mathbf{C}^2/G_j \to [0, \infty)$ denotes the radius function. Due to work of Kronheimer [Kro89a, Kro89b] ALE spaces are very well understood. Here is a summary of his main results.

Theorem 4.1 (Kronheimer)**.** *Let G be a non-trivial finite subgroup of* $\mathrm{SU}(2)$*. Denote by X the manifold underlying the crepant resolution $\widetilde{\mathbf{C}^2/G}$. Then for each three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X, \mathbf{R})$ satisfying*

$$(\alpha_1(\Sigma), \alpha_2(\Sigma), \alpha_3(\Sigma)) \neq 0$$

for each $\Sigma \in H_2(X, \mathbf{Z})$ with $\Sigma \cdot \Sigma = -2$ there is a unique ALE hyperkähler structure on X for which the cohomology classes of the Kähler forms $[\omega_i]$ are given by α_i.

There is always a unique crepant resolution of $\mathbf{C}^2/G$. It can be obtained by a sequence of blow-ups. The exceptional divisor E of $X = \widetilde{\mathbf{C}^2/G}$ has irreducible components $\Sigma_1, \dots, \Sigma_k$. By the McKay correspondence [McK80], these components form a basis of $H_2(X, \mathbf{Z})$ and the matrix with coefficients $C_{ij} = -\Sigma_i \cdot \Sigma_j$ is the Cartan matrix associated with the Dynkin diagram corresponding to G in the ADE classification of finite subgroups of $\mathrm{SU}(2)$.

As a consequence of the above, one can always find an ALE space X_j asymptotic to $\mathbf{C}^2/G_j$. If H_j is non-trivial, then we also require an isometric action ρ_j of H_j on X_j which is asymptotic to the action of H_j on $\mathbf{C}^2/G_j$. A collection $\{(X_j, \rho_j)\}$ of this kind is called *resolution data* for Y_0.

Denote by $\pi_j\colon X_j \to \mathbf{C}^2/G_j$ the resolution map for X_j. Let $\pi_{j,t} := t\pi_j : X_j \to \mathbf{C}^2/G_j$ and let

$$\tilde{T}_{j,t} := (T^3 \times \pi_{j,t}^{-1}(B^4_\zeta/G_j))/H_j \quad \text{and} \quad \tilde{T}_t := \bigcup_j \tilde{T}_{j,t}.$$

Then using $\pi_{j,t}$ we can replace each T_j in Y_0 by $\tilde{T}_{j,t}$ and thus obtain a compact 7–manifold Y_t. The Betti numbers of Y_t can be read off easily from this construction. It is easy to see that each of the components S_j of the singular set S give rise to a homology class $[S_j] \in H_3(Y_t)$ and that each $\Sigma \in H_2(X_j)$ invariant under H_j yields a homology class $\Sigma \in H_2(Y_t)$. Moreover, one can work out fundamental group $\pi_1(Y_t)$. This is very important, because G_2–manifolds have full holonomy G_2 if and only if their fundamental group is finite.

On $\tilde{T}_{j,t}$ there is a G_2–structure defined by

$$\tilde{\phi}_t = \delta_1 \wedge \delta_2 \wedge \delta_3 + t^2 \delta_1 \wedge \tilde{\omega}_1 + t^2 \delta_2 \wedge \tilde{\omega}_2 - t^2 \delta_3 \wedge \tilde{\omega}_3.$$

If we view $Y_0 \setminus S$ as a subset of Y_t, then $\tilde{\phi}_t$ almost matches up with ϕ_0 in the following sense. Introduce a function $d_t\colon Y_t \to [0,\zeta]$ measuring the distance from the "exceptional set" by

$$d_t(p) = \begin{cases} |\pi_{j,t}(y)|, & p = [(x,y)] \in \tilde{T}_{j,t} \\ \zeta, & p \in Y_t \setminus \tilde{T}_t. \end{cases}$$

Then on sets of the form $\{x \in T_t : d_t(x) \geq \epsilon\}$, we have $\tilde{\phi}_t - \phi_0 = O(t^4)$.

Theorem 4.2 (Joyce)**.** *Let $Y_0 = T^7/\Gamma$ be a flat G_2–orbifold and let $\{(X_j,\rho_j)\}$ be resolution data for Y_0. Then there is a constant $\kappa > 0$ such that for $t \in (0,\kappa)$ there is a torsion-free G_2–structure ϕ_t on the 7-manifold Y_t constructed above satisfying the following conditions. For $\alpha \in (0,1)$ there is a constant $c = c(\alpha) > 0$ such that*

$$\|\phi_t - \tilde{\phi}_t\|_{L^\infty} + t^\alpha[\phi_t - \tilde{\phi}_t]_{C^{0,\alpha}} + t\|\nabla(\phi_t - \tilde{\phi}_t)\|_{L^\infty} \leq ct^{3/2}$$

holds on $\tilde{T}_{j,t}$. Moreover, for every $\epsilon > 0$, there is a constant $c = c(\alpha,\epsilon) > 0$ such that

$$\|\phi_t - \phi_0\|_{C^{0,\alpha}} \leq c(\epsilon)t^{3/2}$$

holds on $\{x \in Y_t : d_t(x) \geq \epsilon\}$.

To prove this, one first writes down a closed 3–form $\psi_t \in \Omega^3(Y_t)$ that interpolates between $\tilde{\phi}_t$ and ϕ_0 in the annulus

$$R_t := \{x \in Y_t : d_t(x) \in [\zeta/4, \zeta/2]\}.$$

This can be done reasonably explicitly using cut-off functions, since

$$t^2(\pi_{j,t})_*\tilde{\omega}_i = \omega_i + \mathrm{d}\varrho_i$$

with $\nabla^k\varrho_i = O(r^{-3-k})$ for $k \geq 0$. If $t > 0$ is sufficiently small this will induce a G_2–structure which is torsion-free on the complement of R_t and has small torsion on R_t. The ansatz $\phi_t = \psi_t + \mathrm{d}\eta_t$ leads to the following non-linear equation

$$\mathrm{d}\Theta(\psi_t + \mathrm{d}\eta_t) = 0.$$

The gauge fixed linearisation of this equation is the Laplacian on 2-forms Δ. As t tends to zero, the properties of Δ degenerate but at the same time ψ_t becomes closer and closer to being torsion-free. By careful scaling considerations Joyce [Joy96, Part I] was able to show that the equation can always be solved provided $t > 0$ is sufficiently small. We should point out that the estimates we give here are rather stronger than those given originally by Joyce. The stated estimate (which is still suboptimal) can be established by a careful analysis of Joyce's construction or, alternatively, by applying some of the ideas in this article to the above problem. The details of this argument will be carried out elsewhere.

5 ASD instantons on ALE spaces

Let Γ be a finite subgroup of $\mathrm{SU}(2)$ and let X be an ALE space asymptotic to $\mathbf{C}^2/\Gamma$. Fix a flat connection θ over S^3/Γ and let E be a G–bundle over X asymptotic at infinity to the bundle underlying θ. We consider the space $\mathscr{A}(E,\theta)$ of connections A on E that are asymptotic at infinity to θ (at a certain rate) and the based gauge

group $\mathscr{G}_0$ of gauge transformations that are asymptotic at infinity to the identity (again at a certain rate). The space

$$M(E,\theta) = \{A \in \mathscr{A}(E,\theta) : F_A^+ = 0\}/\mathscr{G}_0$$

is called the *moduli space of framed ASD instantons on E asymptotic to θ*. It does not depend on the precise choices as long as the rate of decay to θ is not faster that r^{-3} but still fast enough to ensure that $\|F_A\|_{L^2} < \infty$. If $\rho\colon \Gamma \to G$ denotes the monodromy representation of θ, then the group $G_\rho = \{g \in G : g\rho g^{-1} = \rho\}$ acts on $M(E,\theta)$. This can be thought of as the residual action of the group gauge group $\mathscr{G}$ (consisting of gauge transformations with bounded derivative).

Theorem 5.1 (Nakajima [Nak90, Theorem 2.6])**.** *The moduli space $M(E,\theta)$ is a smooth hyperkähler manifold.*

Formally, this can be seen as an infinite-dimensional instance of a hyperkähler reduction [HKLR87]. The space $\mathscr{A}(E,\theta)$ inherits a hyperkähler structure from X and the action of the based gauge group $\mathscr{G}_0$ has a hyperkähler moment map given by $\mu(A) = F_A^+$. To make this rigorous one needs to setup a suitable Kuranishi model for $M(E,\theta)$. This can be done using weighted Sobolev spaces as in [Nak90]. The infinitesimal deformation theory is then governed by the operator $\delta_A\colon\ \Omega^1(X,\mathfrak{g}_E) \to \Omega^0(X,\mathfrak{g}_E) \oplus \Omega^+(X,\mathfrak{g}_E)$ defined by

$$\delta_A(a) = (\mathrm{d}_A^* a, \mathrm{d}_A^+ a).$$

Proposition 5.2. *Let X be an ALE space, let E be a G–bundle over X and let $A \in \mathscr{A}(E)$ be a finite energy ASD instanton on E. Then the following holds.*

1. *If $a \in \ker \delta_A$ decays to zero at infinity, then $a = O(r^{-3})$.*
2. *If $(\xi,\omega) \in \ker \delta_A^*$ decays to zero at infinity, then $(\xi,\omega) = 0$.*

The last part of this proposition tells us that the deformation theory is always unobstructed and hence $M(E,\theta)$ is a smooth manifold. By the first part the tangent space of $M(E,\theta)$ at $[A]$ agrees with the L^2 kernel of δ_A and thus the formal hyperkähler structure is indeed well-defined.

Proof of Proposition 5.2. The proof is based on a refined Kato inequality. Since $\delta_A a = 0$, it follows that $|\mathrm{d}|a|| \le \gamma|\nabla_A a|$ with a constant $\gamma < 1$. To see that, recall that the Kato inequality hinges on the Cauchy-Schwarz inequality $|\langle \nabla_A a, a\rangle| \le |\nabla\alpha||\alpha|$. The equation $\delta_A a = 0$ imposes a linear constraint on $\nabla_A a$ incompatible with equality in this estimate unless $\nabla_A a = 0$. This shows that one can find $\gamma < 1$, such that a refined Kato inequality holds. Indeed one can easily compute that $\gamma = \sqrt{3/4}$. Now, the Weitzenböck formula implies that for $\sigma = 2 - 1/\gamma^2 = 2/3$

$$\begin{aligned}
(2/\sigma)\Delta|a|^\sigma &= |a|^{\sigma-2}(\Delta|a|^2 - 2(\sigma-2)|\mathrm{d}|a||^2) \\
&\le |a|^{\sigma-2}(\Delta|a|^2 + 2|\nabla_A a|^2) \\
&= |a|^{\sigma-2}\langle a, \nabla_A^*\nabla_A a\rangle \\
&= |a|^{\sigma-2}(\langle \delta_A^*\delta_A a, a\rangle + \langle\{R,a\},a\rangle + \langle\{F_A,a\},a\rangle) \\
&\le O(r^{-4})|a|^\sigma.
\end{aligned}$$

Here $\{R,.\}$ and $\{F_A,\cdot\}$ denote certain actions of the Riemannian curvature R and the curvature F_A of A respectively. Since we only care about the behaviour of $|a|$ at infinity, we can view it as function on $\mathbf{R}^4$ which satisfies $\Delta_{\mathbf{R}^4}|a|^\sigma \le O(r^{-4})$. Using the analysis in [Joy00, Section 8.3], we can find f on $\mathbf{R}^4$ with $f = O(r^{-2})$ and $\Delta_{\mathbf{R}^4} f = (\Delta_{\mathbf{R}^4}|a|^\sigma)^+$. Now, $g = |a|^\sigma - f$ is a decaying subharmonic function on

$\mathbf{R}^4$, so it decays at least like r^{-2}. This follows from a maximum principle argument using a large multiple of the Green's function as a barrier. Since $\sigma = 2/3$, it follows that $|a|$ decays like r^{-3}. This proves the first part. For the second part, note that $\mathrm{d}_A^*\mathrm{d}_A\xi = 0$. By part one $\mathrm{d}_A\xi = O(r^{-3})$, thus integration by parts yields $\mathrm{d}_A\xi = 0$, hence $\xi = 0$. Similarly, one shows that $\omega = 0$. □

The dimension of $M(E, \theta)$ can be computed using the following index formula which can be proved using the Atiyah-Patodi-Singer index theorem and the Atiyah-Bott-Lefschetz formula to compute the contribution from infinity.

Theorem 5.3 (Nakajima [Nak90, Theorem 2.7])**.** *Let Γ be a non-trivial finite subgroup of $\mathrm{SU}(2)$ and let θ be a flat connection on a G–bundle over S^3/Γ. Let X be an ALE space asymptotic to $\mathbf{C}^2/\Gamma$, let E be a G–bundle over X asymptotic at infinity to the bundle supporting θ and let A be a finite energy ASD instanton on E asymptotic to θ. Then the dimension of the L^2–kernel of δ_A is given by*

$$\dim\ker\delta_A = -2\int_X p_1(\mathfrak{g}_E) + \frac{2}{|\Gamma|}\sum_{g\in\Gamma\setminus\{e\}} \frac{\chi_{\mathfrak{g}}(g) - \dim\mathfrak{g}}{2 - \operatorname{tr} g}.$$

Here $p_1(\mathfrak{g}_E)$ is the Chern-Weil representative of the first Pontryagin class of E and $\chi_{\mathfrak{g}}$ is the character of the representation $\rho\colon \Gamma \to G$ corresponding to θ.

The space $M(E, \theta)$ is usually non-compact and often incomplete. This is related to instantons wandering off to infinity and bubbling phenomena. For details we refer the reader to [Nak90].

Let $\hat{X}$ be a conformal compactification of X. (The point at infinity will be an orbifold singularity, but this causes no trouble). Since $A \in M(E, \theta)$ has finite energy it follows from Uhlenbeck's removable singularities theorem [Uhl82, Theorem 4.1] that A extends to $\hat{X}$. Now radial parallel transport yields a frame at infinity for A in which we can write

$$A = \theta + a \quad \text{with} \quad \nabla^k a = O(r^{-3-k}).$$

This will turn out to be crucial later on. The reader may find it useful to think of $M(E, \theta)$ as a moduli space of ASD instantons on $\hat{X}$ framed at the point at infinity (although equipped with a non-standard metric).

There is a very rich existence theory for ASD instantons on ALE spaces. Gocho–Nakajima [GN92] observed that for each representation $\rho\colon \Gamma \to \mathrm{U}(n)$ there is a bundle $\mathcal{R}_\rho$ over X together with an ASD instanton A_ρ asymptotic to the flat connection determined by ρ, and if σ is a further representation of Γ, then $A_{\rho\oplus\sigma} = A_\rho \oplus A_\sigma$. Kronheimer–Nakajima [KN90] took this as the starting point for an ADHM construction of ASD instantons on ALE spaces. One important consequence of their work is the following rigidity result.

Lemma 5.4 (Kronheimer–Nakajima [KN90, Lemma 7.1])**.** *Each A_ρ is infinitesimally rigid, i.e., the L^2–kernel of δ_{A_ρ} is trivial.*

By combining this result applied to the regular representation with the index formula Kronheimer–Nakajima derive a geometric version of the McKay correspondence [KN90, Appendix A]. Let $\Delta(\Gamma)$ denote the Dynkin diagram associated to Γ in the ADE classification of the finite subgroups of $\mathrm{SU}(2)$. Each vertex of $\Delta(\Gamma)$ corresponds to a non-trivial irreducible representation. We label these by $\rho_1, \ldots, \rho_k$ and denote the associated bundles by $\mathcal{R}_j$ and the associated ASD instantons by A_j.

Theorem 5.5 (Kronheimer–Nakajima)**.** *The harmonic 2–forms $c_1(\mathcal{R}_j) = \frac{i}{2\pi}\operatorname{tr} F_{A_j}$ form a basis of $L^2\mathcal{H}^2(X) \cong H^2(X, \mathbf{R})$ and satisfy*

$$\int_X c_1(\mathcal{R}_i) \wedge c_1(\mathcal{R}_j) = -(C^{-1})_{ij}$$

where C is the Cartan matrix associated with $\Delta(\Gamma)$. Moreover, there is an isometry $\kappa \in \mathrm{Aut}(H_2(X, \mathbf{Z}), \cdot)$ such that $\{c_1(\mathcal{R}_j)\}$ is dual to $\{\kappa[\Sigma_j]\}$, where Σ_j are the irreducible components of the exceptional divisor E of $\widetilde{\mathbf{C}^2/\Gamma}$. If X is isomorphic to $\widetilde{\mathbf{C}^2/\Gamma}$ as a complex manifold, then $\kappa = \mathrm{id}$.

This results is very useful for computing the index of δ_A where A is constructed out of ASD instantons of the form A_ρ (by taking tensor products, direct sums, etc.).

Here is a simple example that we will be used later on.

Proposition 5.6. *Let X be an ALE space asymptotic to $\mathbf{C}^2/\mathbf{Z}_k$. Denote by $\mathcal{R}_j = \mathcal{R}_{\rho_j}$ and $A_j = A_{\rho_j}$ the line bundle and ASD instanton associated with $\rho_j \colon \mathbf{Z}_k \to U(1)$ defined by $\rho_j(\ell) = \exp\left(\frac{2\pi i}{k} j\ell\right)$. For $n, m \in \mathbf{Z}_k$ let $A_{n,m}$ be the induced ASD instanton on the $\mathrm{SO}(3)$–bundle $E_{n,m} = \mathbf{R} \oplus (\mathcal{R}_n^* \otimes \mathcal{R}_{n+m})$. Then $A_{n,m}$ is infinitesimally rigid, asymptotic at infinity to the flat connection over $S^3/\mathbf{Z}_k$ induced by ρ_j,*

$$\frac{1}{8\pi^2} \int |F_{A_{n,m}}|^2 = \frac{(k-m)m}{k}$$

and

$$w_2(E_{n,m}) = c_1(\mathcal{R}_{n+m}) - c_1(\mathcal{R}_n) \in H^2(X, \mathbf{Z}_2).$$

Proof. To see that $A_{n,m}$ is infinitesimally rigid apply Lemma 5.4 to $A_n \oplus A_{n+m}$. The statement about the second Stiefel-Whitney class is obvious.

To compute the energy of $A_{n,m}$, it is enough to note that the first term in the index formula is precisely twice the energy and the second term is given by $\left(-\frac{2}{k}\right)$–times

$$-\sum_{g \neq e} \frac{\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}}{2 - \operatorname{tr} g} = \sum_{j=1}^{k-1} \frac{1 - \cos(2\pi m j/k)}{1 - \cos(2\pi j/k)} = (k-m)m.$$

□

6 Approximate G_2–instantons

Let Y_0 be an admissible G_2–orbifold, let $\{(X_j, \rho_j)\}$ be resolution data for Y_0. Let κ be as in Theorem 4.2 and for $t \in (0, \kappa)$ denote by (Y_t, ϕ_t) the G_2 manifold obtained from Joyce's generalised Kummer construction.

Let θ be a flat connection on a G–bundle E_0 over Y_0. Then the monodromy of θ around S_j induces a representation $\mu_j \colon \pi_1(T_j) \cong (\mathbf{Z}^3 \times G_j) \rtimes H_j \to G$ of the orbifold fundamental group of T_j. If all $\mu_j|_{G_j}$ are trivial there is a straightforward way to lift E_0 and θ up to Y_t. In general, more input is required. A collection $((E_0, \theta), \{(x_j, f_j)\}, \{(E_j, A_j, \tilde{\rho}_j, m_j)\})$ consisting of E_0 and θ as above as well for each j the choice of

- a point $x_j \in S_j$ together with a framing $f_j \colon (E_0)_{x_j} \to G$ of E_0 at x_j,
- a G-bundle E_j over X_j together with a framed ASD instanton A_j asymptotic at infinity to the flat connection on S^3/G_j induced by the representation $\mu_j|_{G_j}$,
- a lift $\tilde{\rho}_j$ of the action ρ_j of H_j on X_j to E_j and
- a homomorphism $m_j \colon \mathbf{Z}^3 \to \mathscr{G}(E_j)$

is called *gluing data* compatible with the resolution data $\{(X_j, \rho_j)\}$ if the following compatibility conditions are satisfied:

- The action $\tilde{\rho}_j$ of H_j on E_j preserves A_j and the induced action on the fibre of E_j at infinity is given by $\mu_j|_{H_j}$.
- The action of $\mathbf{Z}^3$ on E_j given by m_j preserves A_j and the induced action on the fibre of E_j at infinity is given by $\mu_j|_{\mathbf{Z}^3}$.
- For all $h \in H_j$ and $g \in \mathbf{Z}^3$ we have $\tilde{\rho}_j(h)m_j(g)\tilde{\rho}_j(h)^{-1} = m_j(hgh^{-1})$.

We should point out here that it is by far not always possible to extend a choice of (E_0, θ) and $\{(E_j, A_j)\}$ to compatible gluing data. This will become clear from the examples in Section 9. With this notation in place we can prove the first part of Theorem 1.1.

Proposition 6.1. *Let $((E_0, \theta), \{(x_j, f_j)\}, \{(E_j, A_j, \tilde{\rho}_j, m_j)\})$ be gluing data compatible with $\{(X_j, \rho_j)\}$. Then for $t \in (0, \kappa)$ one can construct a G–bundle E_t on Y_t together with a connection A_t satisfying*

$$p_1(E_t) = -\sum_j k_j \,\mathrm{PD}[S_j] \quad \textit{with} \quad k_j = \frac{1}{8\pi^2}\int_{X_j} |F_{A_j}|^2$$

and

$$\langle w_2(E_t), \Sigma\rangle = \langle w_2(E_j), \Sigma\rangle$$

for each $\Sigma \in H_2(X_j)^{H_j} \subset H_2(Y_t)$. Here $[S_j] \in H_3(Y_t)$ is the cohomology class induced by the component S_j of the singular set S.

Proof. The choices of $\tilde{\rho}_j$ and m_j define a lift of the action of $\mathbf{Z}^3 \rtimes H_j$ on $\mathbf{R}^3 \times X_j$ to the pullback of E_j to $\mathbf{R}^3 \times X_j$. Passing to the quotient yields a G–bundle over $(T^3 \times X_j)/H_j$ which we still denoted by E_j. It follows from the compatibility conditions that the pullback of A_j to $\mathbf{R}^3 \times X_j$ passes to this quotient and defines a connection on E_j which we still denote by A_j.

Fix $t \in (0, \kappa)$. Let $R_{j,t} = R_t \cap \tilde{T}_{j,t}$ with R_t and $\tilde{T}_{j,t}$ defined as in Section 4. By the compatibility conditions the monodromy of A_j along S_j on the fibre at infinity matches up with the monodromy of θ along $E_0|_{S_j}$. Thus, by parallel transport the identification of $(E_0)_{x_j}$ with the fibre at infinity of E_j extends to an identification of $E_0|_{R_{j,t}}$ with $E_j|_{R_{j,t}}$. Patching E_0 and the E_j via this identification yields the bundle E_t.

Under the identification of $E_0|_{R_{j,t}}$ with $E_j|_{R_{j,t}}$, we can write

$$A_j = \theta + a_j \quad \text{with} \quad \nabla^k a_j = t^{2+k} O(d_t^{-3-k})$$

where $d_t\colon Y_t \to [0, \zeta]$ is the function measuring the distance from the exceptional set introduced in Section 4. Fix a smooth non-increasing function $\chi : [0, \zeta] \to [0, 1]$ such that $\chi(s) = 1$ for $s \le \zeta/4$ and $\chi(s) = 0$ for $s \ge \zeta/2$. Set $\chi_t := \chi \circ d_t$. After cutting off A_j to $\theta + \chi_t \cdot a_j$ it can be matched with θ and we obtain the connection A_t on the bundle E_t.

Using Chern-Weil theory one can compute $p_1(E_t)$. The statement about $w_2(E_t)$ follows from the naturality of Stiefel-Whitney classes. ∎

If t is small, then the connection A_t is a close to being a G_2–instanton. In order to make this precise we introduce weighted Hölder norms. It will become more transparent over the course of the next two sections that these are well adapted to the problem at hand. We define weight functions by

$$w_t(x) := t + d_t(x) \quad w_t(x, y) := \min\{w_t(x), w_t(y)\}.$$

For a Hölder exponent $\alpha \in (0,1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$[f]_{C^{0,\alpha}_{\beta,t}} := \sup_{d(x,y)\le w_t(x,y)} w_t(x,y)^{\alpha-\beta}\frac{|f(x)-f(y)|}{d(x,y)^\alpha}$$
$$\|f\|_{L^\infty_{\beta,t}} := \|w_t^{-\beta}f\|_{L^\infty}$$
$$\|f\|_{C^{k,\alpha}_{\beta,t}} := \sum_{j=0}^k \|\nabla^j f\|_{L^\infty_{\beta-j,t}} + [\nabla^j f]_{C^{0,\alpha}_{\beta-j,t}}.$$

Here f is a section of a vector bundle on Y_t equipped with a inner product and a compatible connection. Note that, if $\beta = \beta_1 + \beta_2$, then

$$\|f \cdot g\|_{C^{k,\alpha}_{\beta,t}} \le \|f\|_{C^{k,\alpha}_{\beta_1,t}} \|g\|_{C^{k,\alpha}_{\beta_2,t}}.$$

Also for $\beta > \gamma$ we have

$$\|f\|_{C^{k,\alpha}_{\beta,t}} \le t^{\gamma-\beta}\|f\|_{C^{k,\alpha}_{\gamma,t}}.$$

Proposition 6.2. *Let A_t be as Proposition 6.1 and denote by $\psi_t = \Theta(\phi_t)$ the coassociative calibration on corresponding to ϕ_t. Then for $\alpha \in (0,1)$ and $\beta \ge -4$, there is a constant $c = c(\alpha,\beta) > 0$ such that for all $t \in (0,\kappa)$ the following estimate holds*

$$\|F_{A_t} \wedge \psi_t\|_{C^{0,\alpha}_{\beta,t}} \le c(t^2 + t^{-\beta-\alpha-\frac12}).$$

Proof. On $Y_t \setminus \tilde{T}_t$ the connection A_t is flat. Thus we can focus our attention to $\tilde{T}_{j,t}$. By the definition of A_t we have

$$F_{A_t} = (1-\chi_t)F_{A_j} + \mathrm{d}\chi_t \wedge a_j + \frac{\chi_t^2 - \chi_t}{2}[a_j \wedge a_j].$$

The last two terms in this expression are supported in R_t and of order t^2 in $C^{0,\alpha}$. Since the coassociative calibration $\tilde{\psi}_t = \Theta(\tilde{\phi}_t)$ on the model $\tilde{T}_{j,t}$ is given by

$$\tilde{\psi}_t = \frac12\omega_1\wedge\omega_1 + t^2\delta_2\wedge\delta_3\wedge\tilde{\omega}_1 + t^2\delta_3\wedge\delta_1\wedge\tilde{\omega}_2 - t^2\delta_1\wedge\delta_2\wedge\tilde{\omega}_3$$

and $F_{A_j} \wedge \tilde{\psi}_t = 0$, we are left with estimating the size of $F_{A_j} \wedge (\psi_t - \tilde{\psi}_t)$. But by Theorem 4.2

$$\|\psi_t - \tilde{\psi}_t\|_{C^{0,\alpha}_{0,t}} \le ct^{\frac32-\alpha}.$$

Since $\|F_{A_j}\|_{C^{0,\alpha}_{-4,t}} \le ct^2$, this immediately implies the desired estimate. ∎

7 A model operator on $\mathbf{R}^3 \times$ ALE

In order proof Theorem 1.1 we need to find $a_t \in \Omega^1(Y_t, \mathfrak{g}_{E_t})$ such that

$$F_{A_t+a_t} \wedge \psi_t = 0,$$

where $\psi_t = \Theta(\phi_t)$ is the coassociative calibration on Y_t provided that $t > 0$ is small. As we explained in Section 3, this is equivalent to solving

$$*_t\,(F_{A_t+a_t} \wedge \psi_t) + \mathrm{d}_{A_t+a_t}\xi_t = 0$$

for $\xi_t \in \Omega^0(Y_t, \mathfrak{g}_{E_t})$ and $a_t \in \Omega^1(Y_t, \mathfrak{g}_{E_t})$. Here $*_t$ denotes the Hodge star associated with ϕ_t. There is no loss in additionally requiring the gauge fixing condition $\mathrm{d}_{A_t}^* a_t = 0$. The equation to be solved can then be written as

$$L_t(\xi_t, a_t) + Q_t(\xi_t, a_t) + *_t\,(F_{A_t} \wedge \psi_t) = 0$$

where $L_t = L_{A_t}$ is the linearised operator introduced in Section 3 and the non-linear operator Q_t is given by

$$Q_t(\xi, a) = \frac{1}{2} *_t ([a \wedge a] \wedge \psi_t) + [a, \xi].$$

Once the linearisation L_{A_t} is sufficiently well understood, solving this equation is rather easy.

Away from the exceptional set L_t is essentially equivalent to the deformation operator L_θ associated with the flat connection θ. To gain better understanding of L_t near the exceptional set we introduce the following model. Let $(X, \omega_1, \omega_2, \omega_3)$ be an ALE space and let $\delta_1, \delta_2, \delta_3$ be a triple of constant 1–forms on standard $\mathbf{R}^3$. Then the coassociative calibration ψ corresponding to the induced G_2–structure on $\mathbf{R}^3 \times X$ is given by

$$\psi = \frac{1}{2}\omega_1 \wedge \omega_1 + \delta_1 \wedge \delta_2 \wedge \omega_3 + \delta_2 \wedge \delta_3 \wedge \omega_1 - \delta_3 \wedge \delta_1 \wedge \omega_2.$$

Let E be a G–bundle over X together with an ASD instanton A. Without changing the notation we pullback E and A to $\mathbf{R}^3 \times X$. As in Section 3 we denote by $L_A \colon \Omega^0(\mathbf{R}^3 \times X, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^3 \times X, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X, \mathfrak{g}_E)$ the deformation operator associated with A defined by

$$L_A = \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix}.$$

Separating the parts of L_A which differentiate in the direction of $\mathbf{R}^3$ from those that differentiate in the direction of X yields the following.

Proposition 7.1. *We can write $L_A = F + D_A$ where F and D_A are commuting formally self-adjoint operators and $F^2 = \Delta_{\mathbf{R}^3}$. In particular,*

$$L_A^* L_A = \Delta_{\mathbf{R}^3} + D_A^* D_A.$$

Under the identification $T_x^*\mathbf{R}^3 \to \Lambda^+ T_y^* X$ defined by $\delta_1 \mapsto \omega_1$, $\delta_2 \mapsto \omega_2$, $\delta_3 \mapsto -\omega_3$ the operator D_A takes the form

$$D_A = \begin{pmatrix} 0 & \delta_A \\ \delta_A^* & 0 \end{pmatrix}.$$

To understand the properties of L_A we work with weighted Hölder norms. We define weight functions by

$$w(x) := 1 + |\pi(x)|, \quad w(x,y) := \min\{w(x), w(y)\}.$$

Here $\pi\colon X \to \mathbf{C}^2/G$ denotes the resolution map associated with the ALE space. For a Hölder exponent $\alpha \in (0,1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$[f]_{C^{0,\alpha}_\beta} := \sup_{d(x,y) \le w(x,y)} w(x,y)^{\alpha-\beta} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

$$\|f\|_{L^\infty_\beta} := \|w^{-\beta} f\|_{L^\infty}$$

$$\|f\|_{C^{k,\alpha}_\beta} := \sum_{j=0}^{k} \|\nabla^j f\|_{L^\infty_{\beta-j}} + [\nabla^j f]_{C^{0,\alpha}_{\beta-j}}.$$

Here f is a section of a vector bundle on $\mathbf{R}^3 \times X$ equipped with a inner product and a compatible connection.

These norms are related to the ones introduced in Section 6 as follows. In the situation of Section 6 define $p_{j,t} : \mathbf{R}^3 \times X_j \to \tilde{T}_{j,t}$ by

$$p_t(x, y) = [(tx, y)].$$

This map pulls back the metric on $\tilde{T}_{j,t}$ associated with $\tilde{\phi}_t$, that is $g_{\tilde{\phi}_t} = g_{T^3} \oplus t^2 g_{X_j}$, to $t^2(g_{\mathbf{R}^3} \oplus g_{X_j})$. Since the metric on Y_t induced by ϕ_t is uniformly equivalent to the one induced by $\tilde{\phi}_t$ on $\tilde{T}_t$, this yields the following relation

$$\|p_{j,t}^* \alpha\|_{C^{k,\alpha}_\beta(p_t^{-1}(\tilde{T}_{j,t}))} \sim t^{d+\beta} \|\alpha\|_{C^{k,\alpha}_{\beta,t}(\tilde{T}_{j,t})}$$

for d–forms α. Here $\sim$ means equivalent with a constant independent of t. The relation between L_{A_j} and the deformation operator L_t associated with A_t is as follows. If $(\eta, b) = L_t(\xi, a)$ for $\xi \in \Omega^0(Y_t, \mathfrak{g}_{E_t})$ and $a \in \Omega^1(Y_t, \mathfrak{g}_{E_t})$, then

$$\|t(p_{j,t}^*\eta, t^{-1}p_{j,t}^*b) - L_{A_j}(p_{j,t}^*\xi, t^{-1}p_{j,t}^*a)\|_{C^{0,\alpha}_{\beta-1}} \le ct^2 \left(\|p_{j,t}^*\xi\|_{C^{1,\alpha}_\beta} + \|t^{-1}p_{j,t}^*a\|_{C^{1,\alpha}_\beta}\right)$$

for some constant $c = c(\alpha, \beta) > 0$ independent of t. In this sense L_{A_j} is a model for L_t near the exceptional set.

Proposition 7.2. *If $\beta \in (-3, 0)$, then the kernel of $L_A\colon C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ consists of elements of the L^2–kernel of δ_A.*

This follows immediately from Proposition 5.2 and the following lemma which we will prove in Appendix A.

Lemma 7.3. *Let E be a vector bundle of bounded geometry over a Riemannian manifold X of bounded geometry and suppose that $D\colon C^\infty(X, E) \to C^\infty(X, E)$ is a bounded uniformly elliptic operator of second order which is positive, i.e., $\langle Da, a\rangle \ge 0$ for all $a \in W^{2,2}(X, E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

and there are $s \in \mathbf{R}$ and $p \in (1, \infty)$ such that $\|a(x, \cdot)\|_{W^{s,p}}$ is bounded independent of $x \in \mathbf{R}^n$, then a is constant in the $\mathbf{R}^n$–direction, that is $a(x, y) = a(y)$.

If A is not infinitesimally rigid, then although the kernel of L_A is finite dimensional, there is an infinite dimensional "effective kernel", that is to say, for each $\epsilon > 0$ there is an infinite dimensional space of elements of the form χa where χ is a smooth function on $\mathbf{R}^3$ and $a \in \ker \delta_A$ such that $\|L_A \chi a\|_{C^{0,\alpha}_{\beta-1}} \le \epsilon \|\chi a\|_{C^{1,\alpha}_\beta}$. This problem can be remedied to some extend by working orthogonal to elements of this form. But it clearly shows, that one cannot expect be able to solve the non-linear equation introduced above in general. On the other hand, if A is infinitesimally rigid, then it follows from Proposition 7.2 that L_A is injective. In fact, under this assumption one can show that L_A is invertible using the following Schauder estimate and the observation that L_A has no "kernel at infinity", which is essentially what will be established in Case 3 of the proof of Proposition 8.5.

Proposition 7.4. *For $\alpha \in (0, 1)$ and $\beta \in \mathbf{R}$ there is a constant $c = c(\alpha, \beta) > 0$, such that the following estimate holds*

$$\|\xi\|_{C^{1,\alpha}_\beta} + \|a\|_{C^{1,\alpha}_\beta} \le c(\|L_A(\xi, a)\|_{C^{0,\alpha}_{\beta-1}} + \|\xi\|_{L^\infty_\beta} + \|a\|_{L^\infty_\beta}).$$

Proof. The desired estimate is local in the sense that is enough to prove estimates of the form

$$\|\xi\|_{C^{1,\alpha}_\beta(U_i)} + \|a\|_{C^{1,\alpha}_\beta(U_i)} \leq c(\|L_A(\xi,a)\|_{C^{0,\alpha}_{\beta-1}} + \|\xi\|_{L^\infty_\beta} + \|a\|_{L^\infty_\beta})$$

with $c > 0$ independent of i, where $\{U_i\}$ is an open cover of $\mathbf{R}^3 \times X$.

Fix $R > 0$ suitably large and set $U_0 = \{(x,y) \in \mathbf{R}^3 \times X : |\pi(x)| \leq R\}$. Then there clearly is a constant $c > 0$ such that the above estimate holds for $U_i = U_0$. Now pick a sequence $(x_i,y_i) \in \mathbf{R}^3 \times X$ such that $r_i := |\pi(y_i)| \geq R$ and the balls $U_i := B_{r_i/8}(x_i,y_i)$ cover the complement of U_0. Now on U_i, we have a Schauder estimate of the form

$$\begin{aligned}\|\underline{a}\|_{L^\infty(U_i)} + r_i^\alpha[\underline{a}]_{C^{0,\alpha}(U_i)} + r_i\|\nabla_A\underline{a}\|_{L^\infty(U_i)} + r_i^{1+\alpha}[\nabla_A\underline{a}]_{C^{0,\alpha}(U_i)} \\ \leq c\left(r_i\|L_A\underline{a}\|_{L^\infty(V_i)} + r_i^{1+\alpha}[L_A\underline{a}]_{C^{0,\alpha}(V_i)} + \|\underline{a}\|_{L^\infty(V_i)}\right)\end{aligned}$$

where $V_i = B_{r_i/4}(x_i,y_i)$ and $\underline{a} = (\xi,a)$. The constant $c > 0$ depends continuously on the coefficients of L_A over V_i and it is thus easy to see that one can find a constant $c > 0$ such that the above estimate holds for all i. Since on V_i we have $\frac{1}{2}r_i \leq w \leq 2r_i$ multiplying the above Schauder estimate by $r_i^{-\beta}$ yields the desired local estimate. □

8 Deforming to genuine G_2–instantons

We continue with the assumptions of Section 6 and we suppose that the connection A_t on E_t over Y_t was constructed using Proposition 6.1 from a choice of compatible gluing data $((E_0,\theta),\{(x_j,f_j)\},\{(E_j,A_j,\tilde{\rho}_j,m_j)\})$. In this section we will prove the following result. This finishes the proof of Theorem 1.1.

Proposition 8.1. *Assume that θ is regular as a G_2–instanton and that each A_j is infinitesimally rigid. Then there is a constant $c > 0$ such that for t sufficiently small there is a small perturbation $a_t \in \Omega^1(Y_t,\mathfrak{g}_{E_t})$ satisfying*

$$F_{A_t+a_t} \wedge \psi_t = 0$$

and $\|a_t\|_{C^{1,1/2}_{-1,t}} \leq ct$. Here $\psi_t = \Theta(\phi_t)$ denotes the coassociative calibration on Y_t. Moreover, $A_t + a_t$ is a regular G_2–instanton.

As discussed in Section 7 it is crucial to understand the properties of the linearisation L_t. The key to proving Proposition 8.1 is the following result.

Proposition 8.2. *Given $\alpha \in (0,1)$ and $\beta \in (-3,0)$, there is a constant $c = c(\alpha,\beta) > 0$ such that for t sufficiently small the following estimate holds*

$$\|\xi\|_{C^{1,\alpha}_{\beta,t}} + \|a\|_{C^{1,\alpha}_{\beta,t}} \leq c\|L_t(\xi,a)\|_{C^{0,\alpha}_{\beta-1,t}}.$$

Before we move on to prove this, let us quickly show how it is used to establish Proposition 8.1. Recall the following elementary observation.

Lemma 8.3. *Let X be a Banach space and let $T\colon X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \leq c(\|x\| + \|y\|)\|x - y\|.$$

Then if $y \in X$ satisfies $\|y\| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $\|x\| \leq \frac{1}{5c}$ solving

$$x + Tx = y.$$

Moreover, the unique solution satisfies $\|x\| \leq 2\|y\|$.

Proof of Proposition 8.1. By Proposition 8.2 L_t injective and has closed range. Since L_t is formally self-adjoint, it follows from elliptic regularity that L_t is also surjective and thus invertible. Denote its inverse by R_t.

If we set $(\xi_t, a_t) = R_t(\eta_t, b_t)$, then the equation we need to solve becomes

$$(\eta_t, b_t) + Q_t(R_t(\eta_t, b_t)) = - *_t (F_{A_t} \wedge \psi_t) .$$

It follows from Proposition 8.2 with $\beta = -1$ and $\alpha = \frac{1}{2}$ that

$$\begin{aligned} &\|Q_t(R_t(\eta_1, b_1)) - Q_t(R_t(\eta_2, b_2))\|_{C^{0,1/2}_{-2,t}} \\ &\quad \le c \left(\|(\eta_1, b_1)\|_{C^{0,1/2}_{-2,t}} + \|(\eta_2, b_2)\|_{C^{0,1/2}_{-2,t}} \right) \|(\eta_1, b_1) - (\eta_2, b_2)\|_{C^{0,1/2}_{-2,t}} \end{aligned}$$

with a constant $c > 0$ independent of t. Since by Proposition 6.2

$$\|F_{A_t} \wedge \psi_t\|_{C^{0,1/2}_{-2,t}} \le ct$$

Lemma 8.3 provides us with a unique solution (η_t, b_t) to the above equation provided t is small enough. Then $(\xi_t, a_t) = R_t(\eta_t, b_t) \in C^{1,1/2}_{-1,t}$ is the desired solution of

$$*_t (F_{A_t + a_t} \wedge \psi_t) + \mathrm{d}_{A_t + a}\xi_t = 0$$

and satisfies $\|a_t\|_{C^{1,1/2}_{-1,t}} \le ct$.

This solution is smooth by elliptic regularity. To see that $A_t + a_t$ is a regular G_2–instanton, note that $\|R_t L_{A_t + a_t} - \mathrm{id}\| \le ct$ and thus $L_{A_t + a_t}$ is invertible for t sufficiently small. ☐

Before embarking on the proof of Proposition 8.2, it will be helpful to make a few observations. On $Y_t \setminus \tilde{T}_t$ the operators L_t and L_θ agree. For fixed $\epsilon > 0$, the norms introduced in Section 6 considered on $\{x \in Y_t : d_t(x) \ge \epsilon > 0\}$ are uniformly equivalent to the corresponding unweighted Hölder norms. Since A_j is asymptotic to θ, for any given $\epsilon > 0$ the restriction of L_t to $\{x \in Y_t : d_t(x) > \epsilon\}$ becomes arbitrarily close to L_θ restricted to $\{x \in Y_0 : d(x, S) > \epsilon\}$ as t goes to zero. These observations combined with remarks preceding Proposition 7.2 yield the following Schauder estimate.

Proposition 8.4. *For $\alpha \in (0,1)$ and $\beta \in \mathbf{R}$ there is a constant $c = c(\alpha, \beta) > 0$ such that the following estimate holds for*

$$\|a\|_{C^{1,\alpha}_{\beta,t}} + \|\xi\|_{C^{1,\alpha}_{\beta,t}} \le c(\|L_t(a, \xi)\|_{C^{0,\alpha}_{\beta-1,t}} + \|a\|_{L^\infty_{\beta,t}} + \|\xi\|_{L^\infty_{\beta,t}}).$$

This reduces the proof of Proposition 8.2 to the following statement.

Proposition 8.5. *For $\alpha \in (0,1)$ and $\beta \in (-3, 0)$ there is a constant $c = c(\alpha, \beta) > 0$ such that the following estimate holds for t sufficiently small*

$$\|\xi\|_{L^\infty_{\beta,t}} + \|a\|_{L^\infty_{\beta,t}} \le c\|L_t(\xi, a)\|_{C^{0,\alpha}_{\beta-1,t}}.$$

Proof. Suppose not. Then there exists a sequence (ξ_i, a_i) and a null-sequence t_i such that

$$\|a_i\|_{L^\infty_{\beta,t_i}} + \|\xi_i\|_{L^\infty_{\beta,t_i}} = 1 \quad \text{and} \quad \|L_{t_i}(a, \xi)\|_{C^{0,\alpha}_{\beta-1,t_i}} \le \frac{1}{i}.$$

Hence by Proposition 8.4

$$\|\xi_i\|_{C^{1,\alpha}_{\beta,t_i}} + \|a\|_{C^{1,\alpha}_{\beta,t_i}} \le 2c.$$

Pick $x_i \in Y_{t_i}$ such that

$$w_{t_i}(x_i)^{-\beta}(|\xi_i(x_i)| + |a_i(x_i)|) = 1.$$

Without loss of generality one of the following three cases occurs. We will rule out all of them, thus proving the proposition.

Case 1. *The sequence x_i accumulates on the regular part of Y_0:* $\lim d_{t_i}(x_i) > 0$.

Let K be a compact subset of $Y_0 \setminus S$. We can view K as a subset of Y_t. As t goes to zero, the metric on K induced from the metric on Y_t converges to the metric on Y_0, similarly we can identify $E_0|_K$ with $E_t|_K$ and via this identification A_t converges to θ on K. When restricted to K, ξ_i and a_i are uniformly bounded in $C^{1,\alpha}$. Thus, by Arzelà-Ascoli, we can assume (after passing to a subsequence) that (ξ_i, a_i) converges to a limit (ξ, a) in $C^{1,\alpha/2}$. Since K was arbitrary this yields $(\xi, a) \in \Omega^0(Y_0 \setminus S, \mathfrak{g}_{E_0}) \oplus \Omega^1(Y_0 \setminus S, \mathfrak{g}_{E_0})$ satisfying

$$|\xi| + |a| < c \cdot d(\cdot, S)^\beta \quad \text{and} \quad |\nabla_\theta \xi| + |\nabla_\theta a| < c \cdot d(\cdot, S)^{\beta-1}$$

as well as

$$L_\theta(\xi, a) = 0.$$

Since $\beta > -3$, the first condition implies that (ξ, a) satisfies $L_\theta(\xi, a) = 0$ in the sense of distributions on all of Y_0. But then elliptic regularity implies that (ξ, a) is smooth. Because θ is assumed to be regular (ξ, a) must be zero. But, we can also assume that x_i converges to $x \in Y_0 \setminus S$, and thus $|\xi|(x) + |a|(x) = d(x, S)^\beta \neq 0$. This is a contradiction.

Case 2. *The sequence x_i accumulates on one of the ALE spaces:* $\lim d_{t_i}(x_i)/t_i < \infty$.

We can assume that x_i converges to $x \in S_j$ for some j. There is no loss in assuming that each x_i is in $\tilde{T}_{j,t_i}$. Pulling (ξ_i, a_i) and x_i back to $\mathbf{R}^3 \times X$ via $p_{j,t}$ as in the remarks preceding Proposition 7.2 yields $(\tilde{\xi}_i, \tilde{a}_i) \in \Omega^0(\mathbf{R}^3 \times X_j, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X_j, \mathfrak{g}_E)$ as well as $\tilde{x}_i \in \mathbf{R}^3 \times X_j$ satisfying

$$\|\tilde{\xi}_i\|_{C^{1,\alpha}_\beta} + \|\tilde{a}_i\|_{C^{1,\alpha}_\beta} \leq 4c$$
$$(1 + |\pi_j(\tilde{x}_i)|)^{-\beta} \left(|\tilde{\xi}_i(\tilde{x}_i)| + |\tilde{a}(\tilde{x}_i)|\right) \geq \frac{1}{2}$$
$$\|L_{A_j}(\tilde{\xi}_i, \tilde{a}_i)\|_{C^{0,\alpha}_{\beta-1}} \leq 2/i.$$

There is a slight abuse of notation here in that $(\tilde{\xi}_i, \tilde{a}_i)$ are not really defined over all of $\mathbf{R}^3 \times X_j$ but just over an exhausting sequence of subsets. This does not cause any trouble.

Again we can assume that there is $(\tilde{\xi}, \tilde{a}) \in \Omega^0(\mathbf{R}^3 \times X_j, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X_j, \mathfrak{g}_E))$ such that $(\tilde{\xi}_i, \tilde{a}_i)$ converges to $(\tilde{\xi}, \tilde{a})$ in $C^{1,\alpha/2}$ on compact subsets of $\mathbf{R}^3 \times X_j$. It follows that $(\tilde{\xi}, \tilde{a}) \in C^{1,\alpha/2}_\beta$ satisfies

$$L_{A_j}(\tilde{\xi}, \tilde{a}) = 0.$$

But then it must be zero by Proposition 7.2, since $\beta \in (-3, 0)$ and each A_j is infinitesimally rigid. On the other hand by translation we can always arrange that the $\mathbf{R}^3$–component of $\tilde{x}_i$ is zero and thus we can view $\tilde{x}_i$ as a point in X_j. Then the condition $\lim d_{t_i}(x_i)/t_i < \infty$ translates into $\lim |\pi_j(\tilde{x}_i)| < \infty$. Therefore, we can assume without loss of generality that $\tilde{x}_i$ converges to some point $\tilde{x} \in X_j$. But then $|\tilde{\xi}(\tilde{x})| + |\tilde{a}(\tilde{x})| \geq \frac{1}{2}(1 + |\pi_j(\tilde{x})|)^\beta > 0$, which contradicts $(\tilde{\xi}, \tilde{a}) = 0$.

Case 3. *The sequence x_i accumulates on one of the necks:* $\lim d_{t_i}(x_i) = 0$ *and* $\lim d_{t_i}(x_i)/t_i = \infty$.

As in Case 2, we rescale to obtain $(\tilde{\xi}_i, \tilde{a}_i)$ and $\tilde{x}_i$. Again we can assume by translation that the $\mathbf{R}^3$–component of $\tilde{x}_i$ is zero. Since $\lim d_{t_i}(x_i)/t_i = \infty$, we have $\lim |\pi_j(\tilde{x}_i)| = \infty$. Now fix a sequence $R_i > 0$ going to infinity such that $R_i/|\pi_j(\tilde{x}_i)|$ goes to zero. We can think of the sets $\mathbf{R}^3 \times (\mathbf{C}^2 \setminus B^4_{R_i})/G_j$ as subsets of $\mathbf{R}^3 \times X_j$. Restricting $(\tilde{\xi}, \tilde{a}_i)$ to this set and rescaling by $1/|\pi_j(\tilde{x}_i)|$ yields (with a small abuse of notation and without relabelling) a sequence $(\tilde{\xi}_i, \tilde{a}_i) \in \Omega^0(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus \{0\})/G_j) \oplus \Omega^1(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus \{0\})/G_j)$ and $\tilde{x}_i \in \mathbf{C}^2 \setminus \{0\}$ satisfying

$$\|\tilde{\xi}_i\|_{C^{1,\alpha}_{\beta,0}} + \|\tilde{a}_i\|_{C^{1,\alpha}_{\beta,0}} \le 8c$$
$$|\tilde{x}_j|^{-\beta}(|\tilde{\xi}_i(\tilde{x}_i)| + |\tilde{a}_i(\tilde{x}_i)|) \ge \frac{1}{4}$$
$$\|L(\tilde{\xi}_i, \tilde{a}_i)\|_{C^{0,\alpha}_{\beta-1,0}} \le 4/i.$$

Here the norms $\|\cdot\|_{C^{k,\alpha}_{\beta,0}}$ are defined like those in Section 7 except with the weight function now defined by $w(x, y) = |y|$ for $(x, y) \in \mathbf{R}^3 \times \mathbf{C}^2/G_j$. The operator L is defined by

$$L(\xi, a) = (\mathrm{d}^* a, \mathrm{d}\xi + *(\psi \wedge \mathrm{d}a))$$

where $\psi = \frac{1}{2}\omega_1 \wedge \omega_1 + \delta_2 \wedge \delta_3 \wedge \omega_1 + \delta_3 \wedge \delta_1 \wedge \omega_2 - \delta_1 \wedge \delta_2 \wedge \omega_3$ with $\omega_i \in \Omega^2(\mathbf{C}^2)$ as given in Section 4.

Passing to a limit via Arzelà-Ascoli as before yields $(\tilde{\xi}, \tilde{a}) \in \Omega^0(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus \{0\})/G_j) \oplus \Omega^1(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus \{0\})/G_j)$ satisfying

$$|\tilde{\xi}| + |\tilde{a}| < cr^{\beta}, \quad |\nabla\tilde{\xi}| + |\nabla\tilde{a}| < cr^{\beta-1},$$

where $r\colon \mathbf{C}^2/G_j \to [0, \infty)$ is the radius function, and

$$L(\tilde{\xi}, \tilde{a}) = 0.$$

Again, since $\beta > -3$, $L(\tilde{\xi}, \tilde{a}) = 0$ actually holds in the sense of distributions on all of $\mathbf{R}^3 \times \mathbf{C}^2/G_j$. But $L^*L = \Delta_{\mathbf{R}^3} + \Delta_{\mathbf{C}^2}$, so Lemma 7.3 shows that $(\tilde{\xi}, \tilde{a})$ is invariant in the $\mathbf{R}^3$–direction. Thus we can think of the components of $(\tilde{\xi}, \tilde{a})$ as harmonic functions on $\mathbf{C}^2$. Since $\beta < 0$, they decay to zero at infinity and thus vanish identically. But we know that $|\tilde{x}_i| = 1$ and thus (without loss of generality) $\tilde{x}_i$ converges to a point $\tilde{x} \in \mathbf{C}^2/G_j$ with $|\tilde{x}| = 1$. At this point $|\tilde{\xi}|(\tilde{x}) + |\tilde{a}|(\tilde{x}) \ge \frac{1}{4}$, contradicting $(\tilde{\xi}, \tilde{a}) = 0$. □

9 Examples with $G = \mathrm{SO}(3)$

We construct a few examples of G_2–instantons on the G_2-manifolds from [Joy00, Section 12.3 and 12.4]. Consider the flat G_2–structure on T^7 given by

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}.$$

Here $\mathrm{d}x^{ijk}$ is a shorthand for $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ and $x_1, \ldots, x_7$ are standard coordinates on $T^7 = \mathbf{R}^7/\mathbf{Z}^7$. The G_2–structure ϕ_0 is preserved by $\alpha, \beta, \gamma \in \mathrm{Diff}(T^7)$ defined by

$$\alpha(x_1, \ldots, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)$$
$$\beta(x_1, \ldots, x_7) = \left(x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7\right)$$
$$\gamma(x_1, \ldots, x_7) = \left(-x_1, x_2, -x_3, x_4, -x_5, x_6, \tfrac{1}{2} - x_7\right).$$

It is easy to see that $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbf{Z}_2^3$.

To understand the singular set S of T^7/Γ note that the only elements of Γ having fixed points are α, β, γ. The fixed point set of each of these elements consists of 16 copies of T^3. The group $\langle \beta, \gamma \rangle$ acts freely on the set of T^3 fixed by α and $\langle \alpha, \gamma \rangle$ acts freely on the set of T^3 fixed by β while $\alpha\beta \in \langle \alpha, \beta \rangle$ acts trivially on the set of T^3 fixed by γ. It follows that S consists of 8 copies of T^3 coming from the fixed points of α and β and 8 copies of $T^3/\mathbf{Z}_2$. Near the copies of T^3 the singular set is modelled on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$ while near the copies of $T^3/\mathbf{Z}_2$ it is modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}_2)/\mathbf{Z}_2$ where the action of $\mathbf{Z}_2$ on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$ is given by

$$(x_1, x_2, x_3, \pm(z_1, z_2)) \mapsto (x_1, x_2, x_3 + \tfrac{1}{2}, \pm(z_1, -z_2)).$$

The 8 copies of T^3 can be desingularised by any choice of 8 ALE spaces asymptotic to $\mathbf{C}^2/\mathbf{Z}^2$. To desingularise the copies of $T^3/\mathbf{Z}^2$ we need to chose ALE spaces which admit an isometric action of $\mathbf{Z}^2$ asymptotic to the action $\mathbf{Z}^2$ on $\mathbf{C}^2/\mathbf{Z}^2$ given by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$. Two possible choices are the resolution of $\mathbf{C}^2/\mathbf{Z}^2$ or a smoothing of $\mathbf{C}^2/\mathbf{Z}^2$. See [Joy00, pp. 313–314] for details.

We construct our examples on desingularisations of quotients of T^7/Γ. To this end we define $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Diff}(T^7)$ by

$$\begin{aligned}
\sigma_1(x_1, \ldots, x_7) &= \left(x_1, x_2, \tfrac{1}{2} + x_3, \tfrac{1}{2} + x_4, \tfrac{1}{2} + x_5, x_6, x_7\right) \\
\sigma_2(x_1, \ldots, x_7) &= \left(x_1, \tfrac{1}{2} + x_2, x_3, \tfrac{1}{2} + x_4, x_5, x_6, x_7\right) \\
\sigma_3(x_1, \ldots, x_7) &= \left(\tfrac{1}{2} + x_1, x_2, x_3, x_4, \tfrac{1}{2} + x_5, \tfrac{1}{2} + x_6, x_7\right).
\end{aligned}$$

The elements σ_j commute with all elements of Γ and thus act on T^7/Γ. Moreover, the action is free.

Example 9.1. Let $A = \langle \sigma_2, \sigma_3 \rangle$. By analysing how A acts on the singular set of T^7/Γ one can see that the singular set of $Y_0 = T^7/(\Gamma \times A)$ consists of one copy of T^3, denoted by S_1, and 6 copies of $T^3/\mathbf{Z}_2$, denoted by $S_2, \ldots, S_7$. S_1 has a neighbourhood modelled on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$ while $S_2, \ldots, S_6$ have neighbourhoods modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}^2)/\mathbf{Z}^2$ where $\mathbf{Z}_2$ acts freely on T^3 and by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$ on $\mathbf{C}^2/\mathbf{Z}^2$. As before, S_1 can be desingularised by a choice of any ALE space asymptotic to $\mathbf{C}^2/\mathbf{Z}_2$. $S_2 \ldots, S_6$ can be desingularised by the resolution of $\mathbf{C}^2/\mathbf{Z}_2$ or a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$.

To compute the orbifold fundamental group $\pi_1(Y_0)$, note that it is isomorphic to the fundamental group $\pi_1(Y_0 \setminus S)$ of the regular part of Y_0. Denote by $p\colon R^7 \to Y_0$ the canonical projection. Then $p\colon p^{-1}(Y_0 \setminus S) \to Y_0 \setminus S$ is a universal cover. Up to conjugation, we can therefore identify $\pi_1(Y_0)$ with the group of deck transformations:

$$\pi_1(Y_0) = \langle \alpha, \beta, \gamma, \sigma_2, \sigma_3, \tau_1, \ldots, \tau_7 \rangle \subset \mathrm{GL}(7).$$

Here we think of $\alpha, \beta, \gamma, \sigma_2, \sigma_3$ as elements of GL(7) defined by the formulae above and τ_i translates the i–th coordinate of $\mathbf{R}^7$ by 1. The group $\pi_1(Y_0)$ is a non-split extension

$$0 \to \mathbf{Z}^7 \to \pi_1(Y_0) \to \Gamma \times A \to 0.$$

To work out the orbifold fundamental $\pi_1(T_j)$ of T_j, again up to conjugation, one simply has to understand the subgroup of deck transformations preserving a fixed component of $p^{-1}(T_j) \subset p^{-1}(Y_0 \setminus S)$. In this way one can compute

$$\begin{gathered}
\pi_1(T_1) = \langle \alpha, \tau_1, \tau_2, \tau_3 \rangle \\
\pi_1(T_2) = \langle \beta, \sigma_3\alpha, \tau_1, \tau_4, \tau_5 \rangle \quad \pi_1(T_3) = \langle \tau_3\beta, \sigma_3\alpha, \tau_1, \tau_4, \tau_5 \rangle \\
\pi_1(T_4) = \langle \gamma, \alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle \quad \pi_1(T_5) = \langle \tau_3\gamma, \tau_3\alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle \\
\pi_1(T_6) = \langle \tau_5\gamma, \tau_5\alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle \quad \pi_1(T_7) = \langle \tau_3\tau_5\gamma, \tau_3\tau_5\alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle.
\end{gathered}$$

Here τ_2 does not appear explicitly in $\pi_1(T_j)$, for $j = 4, \dots, 7$, because $\sigma_2^2 = \tau_2\tau_4$.

Denote by $V = \langle a, b, c \,|\, a^2 = b^2 = c^2 = 1, ab = c \rangle = \mathbf{Z}_2^2$ the Klein four-group. V can be thought of as a subgroup of $\mathrm{SO}(3)$. We define $\rho \colon \pi_1(Y_0) \to V \subset \mathrm{SO}(3)$ by

$$\begin{gathered} \beta, \gamma, \tau_1, \dots, \tau_7 \mapsto 1 \\ \alpha \mapsto a \quad \sigma_2 \mapsto a \quad \sigma_3 \mapsto b. \end{gathered}$$

To see that the flat connection θ induced by ρ is regular as a G_2–instanton we use the following observation.

Proposition 9.2. *A flat connection θ on a G–bundle E_0 over a flat G_2–orbifold Y_0 corresponding to a representation $\rho \colon \pi_1(Y_0) \to \mathrm{G}$ is regular as a G_2–instanton if and only if the induced representation of $\pi_1(Y_0)$ on $\mathfrak{g} \oplus (\mathbf{R}^7 \otimes \mathfrak{g})$ has no non-zero fixed vectors.*

Proof. Since Y_0 is flat as a Riemannian orbifold and θ is a flat connection the Weitzenböck formula takes the form

$$L_\theta^* L_\theta = \nabla_\theta^* \nabla_\theta.$$

Therefore, all infinitesimal deformations of θ are actually parallel sections of the bundle $\mathfrak{g}_{E_0} \oplus (T^*Y_0 \otimes \mathfrak{g}_{E_0})$ and these are in one-to-one correspondence with fixed vectors of the representation of $\pi_1(Y_0)$ on $\mathfrak{g} \oplus \mathbf{R}^7 \otimes \mathfrak{g}$. □

The elements σ_2 and σ_3 act trivially on $\mathbf{R}^7$ and their action on $\mathfrak{so}(3)$ has no common non-zero fixed vectors. Therefore the action of $\pi_1(Y_0)$ on $\mathfrak{g} \oplus \mathbf{R}^7 \otimes \mathfrak{g}$ has no non-zero fixed vector and thus θ is regular.

The monodromy representation $\mu_j|_{G_j} \colon G_j = \mathbf{Z}_2 \to \mathrm{SO}(3)$ associated with the flat connection θ is non-trivial only for $j = 1$. Let $A_1 = A_{0,1}$ be the infinitesimally rigid ASD instanton on $E_1 = E_{0,1}$ given in Proposition 5.6. For $j = 2, \dots, 6$, we choose A_j to be the product connection on the trivial bundle E_j. It is easy to see that this can be extended to a collection of compatible gluing data independent of how the S_j are desingularised. Thus we obtain examples of G_2–instantons on each of these desingularisations by appealing to Theorem 1.1.

Note that any choice of resolution data for $T^7/(\Gamma \times A)$ lifts to an A–invariant choice of resolution data for T^7/Γ. We can then carry out Joyce's generalised Kummer construction in a A–invariant way and lift up the G_2–instanton constructed above. But we could have not constructed this G_2-instanton directly using Theorem 1.1, since the lift of θ to T^7/Γ is not regular.

Example 9.3. Here is a more complicated example. Let $Y_0 = T^7/(\Gamma \times A)$ be as before. Define $\rho \colon \pi_1(Y_0) \to V \subset \mathrm{SO}(3)$ by

$$\begin{gathered} \gamma, \tau_1, \dots, \tau_7 \mapsto 1 \\ \alpha \mapsto a \quad \beta \mapsto b \quad \sigma_2 \mapsto b \quad \sigma_3 \mapsto a. \end{gathered}$$

Again, the resulting flat connection θ is regular. For $j = 1, 2, 3$, let $A_j = A_{0,1}$ be the rigid ASD instanton on $E_j = E_{0,1}$. For $j = 4, \dots, 7$, let A_j be the product connection on the trivial bundle E_j. To be able to extend this to compatible gluing data we need a lift $\tilde{\rho}_j$ of the action of $\mathbf{Z}_2$ on X_j to E_j preserving A_j and acting trivially on the fibre at infinity for $j = 2, 3$. If X_j is a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$, then the $\mathbf{Z}_2$ action on X_j does lift to E_j preserving A_j. But the action does not lift if X_j is the resolution of $\mathbf{C}^2/\mathbf{Z}_2$. The reason for this is that in the first case the action of $\mathbf{Z}_2$ on $H^2(X, \mathbf{R})$ is given by the identity, while in the second case it acts via multiplication by -1, c.f. [Joy00, pp. 313–314]. Thus we can only find compatible gluing data if we resolve both S_2 and S_3 using a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$.

Here is a small modification of this example. Define $\rho\colon\ pi_1(Y_0) \to V \subset \mathrm{SO}(3)$ by

$$\gamma, \tau_1, \dots, \tau_7 \mapsto 1$$
$$\alpha \mapsto a \quad \beta \mapsto b \quad \sigma_2 \mapsto b \quad \sigma_3 \mapsto c.$$

To find compatible gluing data, one simply has to compose $\tilde{\rho}_j$ as above with multiplication by $b \in V \subset \mathscr{G}(E_j)$, for $j = 2, 3$.

Example 9.4. We provide one more example. This is mainly to give the reader something to play with. Also, it further illustrates the phenomenon we already observed in the first example.

Let $B = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, and $Y_0 = T^7/(\Gamma \times B)$. Then the singular set of Y_0 consists of 4 copies of $T^3/\mathbf{Z}^2$, denoted by $S_1, \dots, S_4$, each of which has a neighbourhood modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}^2)/\mathbf{Z}^2$ where $\mathbf{Z}_2$ acts freely on T^3 and by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$ on $\mathbf{C}^2/\mathbf{Z}^2$. The orbifold fundamental group $\pi_1(Y_0)$ is given by

$$\pi_1(Y_0) = \langle \alpha, \beta, \gamma, \sigma_1, \sigma_2, \sigma_3, \tau_1, \dots, \tau_7 \rangle \subset \mathrm{GL}(7).$$

Up to conjugation the fundamental groups of the neighbourhoods T_j of S_j are given by

$$\pi_1(T_1) = \left\langle \alpha, \tau_4^{-1}\tau_5^{-1}\beta\sigma_1\sigma_2\sigma_3, \tau_1, \tau_2, \tau_3 \right\rangle \quad \pi_1(T_2) = \langle \beta, \sigma_3\alpha, \tau_1, \tau_4, \tau_5 \rangle$$
$$\pi_1(T_4) = \langle \gamma, \alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle \quad \pi_1(T_5) = \langle \tau_3\gamma, \tau_3\alpha\beta, \sigma_2, \tau_4, \tau_6 \rangle.$$

Define $\rho\colon\ \pi_1(Y_0) \to V \subset \mathrm{SO}(3)$ by

$$\alpha, \beta, \sigma_3, \tau_1, \dots, \tau_7 \mapsto 1,$$
$$\sigma_1 \mapsto a \quad \sigma_2 \mapsto b \quad \gamma \mapsto b.$$

The induced flat connection θ is clearly regular. If both S_3 and S_4 are desingularised using a resolution of $\mathbf{C}^2/\mathbf{Z}_2$ one can find compatible gluing data.

The resulting G_2–instanton can be lifted to appropriate σ_1–invariant desingularisations of $T^7/(\Gamma \times A)$, but we could not have constructed it there directly, since the lift of θ to $T^7/(\Gamma \times A)$ it is not regular.

The list of examples is far from exhaustive. We invite the reader to try to produce further examples in order to develop a good understanding for when compatible gluing data for a given flat connection can be found and when not.

10 Discussion

Assuming certain restrictions on the first Pontryagin class and the second Stiefel-Whitney class of the underlying bundle we expect that all G_2–instantons on generalised Kummer constructions arise from (a suitable generalisation) of our construction. In these cases one could hope to make the G_2 Casson invariant rigorously defined and, in fact, computable in terms of algebraic data.

If the G_2 Casson invariant can be rigorously defined, it is natural to ask for applications. One (possibly naive) hope is that it will provide a means of distinguishing G_2–manifolds. It is known that there are large sets of input for Joyce's generalised Kummer construction which yield G_2–manifolds that cannot be told apart by simply looking at their fundamental group and their Betti numbers. It is reasonable to expect that not all of those G_2-manifolds are isomorphic. Our work might give some indication as to which of those G_2–manifolds can be distinguished from each other by their G_2 Casson invariant.

Recently, Kovalev-Nordström [KN10] found examples of G_2–manifolds that can be constructed using Joyce's as well as Kovalev's method. It is an interesting question to what extend the images of these constructions in the landscape of G_2–manifolds overlap. Building on Sá Earp's work one could hope to be able better understand the G_2 Casson invariant on G_2–manifolds arising from Kovalev's twisted connected sum construction. This is related to a kind of G_2–analogue of the Atiyah-Floer conjecture. Combining our methods with Sá Earp's might, one day, shed some new light on the relation between Joyce's and Kovalev's construction.

A An infinite dimensional Liouville type theorem

The following result is an abstraction of various results that have appeared in the literature, for example in Pacard-Ritoré's work on the Allen-Cahn equation [PR03, Corollary 7.5] and in Brendle's unpublished work on the Yang–Mills equation in higher dimension [Bre03, Proposition 3.3].

Lemma A.1. *Let E be a vector bundle of bounded geometry over a Riemannian manifold X of bounded geometry and suppose that $D\colon C^\infty(X,E) \to C^\infty(X,E)$ is a bounded uniformly elliptic operator of second order which is positive, i.e., $\langle Da, a\rangle \geq 0$ for all $a \in W^{2,2}(X,E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

and there are $s \in \mathbf{R}$ and $p \in (1,\infty)$ such that $\|a(x,\cdot)\|_{W^{s,p}}$ is bounded independent of $x \in \mathbf{R}^n$, then a is constant in the $\mathbf{R}^n$–direction, that is $a(x,y) = a(y)$.

Note that in the first condition, we allow for s to be negative, and thus the requirement is quite weak.

Here is a heuristic argument. Denote by $\hat{a}$ the partial Fourier transform of a in the $\mathbf{R}^n$–direction. Then $\hat{a}$ solves $(D + |k|^2)\hat{a} = 0$. But $D + |k|^2$ is invertible for $k \neq 0$. Thus $\hat{a}$ is supported on $\{0\} \times X$ and hence must by a linear combination of ($\Gamma(E)$–valued) derivatives of various orders of the δ–function. Reversing the Fourier transform shows that a must be a polynomial in $\mathbf{R}^n$. But then it follows from the assumptions that a is constant in the $\mathbf{R}^n$–direction. Our actual proof will be slightly more pedestrian. First of all we need the following.

Proposition A.2. *Let $s \in \mathbf{R}$ and $p \in (1,\infty)$. Then for $k > 0$ the operator $D + k^2\colon W^{s+2,p}(X,E) \to W^{s,p}(X,E)$ is a an isomorphism. If we fix $k_0 > 0$, then there is a constant $c = c(s,p,k_0) > 0$ such that*

$$\|a\|_{W^{s+2,p}} \leq c\|(D+k^2)a\|_{W^{s,p}}$$

for all $k \geq k_0$. Moreover, the family of operators $(D+k^2)^{-1}$ varies smoothly and satisfies $\|\partial_k^\ell (D+k^2)^{-1}a\|_{W^{s+2,p}} \leq c_\ell(1+k)^\ell \|a\|_{W^{s-2\ell,2}}$ where $c_\ell = c_\ell(s,p,k_0) > 0$.

Proof. We consider the case $s = 0$ and $p = 2$ first. By standard elliptic theory we have $\|a\|_{W^{2,2}} \leq c(\|Da\|_{L^2} + \|a\|_{L^2})$. Combining this with $\|Da\|_{L^2} \leq \|(D+k^2)a\|_{L^2}$ and $k^2\|a\|^2 \leq \big\langle (D+k^2)a, a\big\rangle \leq \|(D+k^2)a\|_{L^2}\|a\|_{L^2}$ one obtains the estimate. This implies that $D + k^2\colon W^{2,2} \to L^2$ is an injective operator with closed range. It is also surjective, since its co-kernel can be identified with the L^2–kernel of $D + k^2$ which is also trivial.

We handle the general case by an argument which the author learned from Haydys [Hay11a, Lemma 2.4]. It is based on the following observations about fractional Sobolev spaces which are easily derived from standard results, c.f. [Tay11, Chapter 13 Section 6], and the assumption that E and X have bounded geometry:

1. For $s, t \in \mathbf{R}$ and $p \in (1, \infty)$ all compactly supported a satisfy

$$\|a\|_{W^{s+2,p}} \leq c(\|(D + k^2)a\|_{W^{s,p}} + \|a\|_{W^{t,p}})$$

for some constant $c = c(s, t, p) > 0$ independent of k. The independence of k essentially follows from the fact that $\|\nabla^2 f\|_{L^p} \leq c\|(\Delta + k^2)f\|_{L^p}$ with c independent of k.

2. The Sobolev embedding $W^{s+t,p} \hookrightarrow W^{s,np/(n-tp)}$ holds for all $s, t \in \mathbf{R}$ and $p \in (0, \infty)$ as long as $n > tp$.

Now, for $s > 0$, we have

$$\|a\|_{W^{s+2,2}} \leq c(\|(D + k^2)a\|_{W^{s,2}} + \|a\|_{L^2}) \leq c\|(D + k^2)a\|_{W^{s,2}}.$$

Which shows that $D + k^2\colon W^{s+2,2} \to W^{s,2}$ is an injective Fredholm operator. It is also surjective, since for $a \in W^{s,2} \subset L^2$ one can easily see that $(D + k^2)^{-1}a \in W^{2,2}$ is in fact in $W^{s+2,2}$. By duality it follows that $D + k^2$ is an isomorphism from $W^{s+2,2}$ to $W^{s,2}$ for $s \in (-\infty, -2)$ with an inverse bounded by a constant c. The remaining range can be covered using the fact that $D + k^2\colon L^2 \to W^{-2,2}$ being the dual of $D + k^2\colon W^{2,2} \to L^2$ is an isomorphism with bounded inverse.

The general case is handled as follows. Combine

$$\|a\|_{W^{s+2,p}} \leq c(\|(D + k^2)a\|_{W^{s,p}} + \|a\|_{W^{s-2,p}})$$

with

$$\|a\|_{W^{s-2,p}} \leq c\|a\|_{W^{t,2}} \leq c\|(D + k^2)a\|_{W^{t+2,2}} \leq c\|(D + k^2)a\|_{W^{s,p}}$$

for $t = s - 2 + (n/2 - n/p)$. It follows that $D + k^2$ is injective and has closed range. Surjectivity follows as before.

The last statement about the smoothness of $(D + k^2)^{-1}$ is clear. □

The lemma can now be proved similar to the argument used by Brendle in [Bre03, Proposition 3.3]. This is essentially the proof of the ingredients from classical distribution theory used in the heuristic proof adapted to our infinite dimensional setting.

Proof of Lemma A.1. We proceed in 3 steps.

Step 1. *Let $\chi \in \mathscr{S}(\mathbf{R}^n)$ be a fast decaying function such that its Fourier transform $\hat{\chi}$ vanishes in a neighbourhood of zero and let $b \in W^{s,p}(X, E)$. Then there is $a \in \mathscr{S}(\mathbf{R}^n, W^{s+2,p}(X, E))$ such that $(\Delta_{\mathbf{R}^n} + D)a = \chi b$.*

We construct $a \in \mathscr{S}(\mathbf{R}^n, W^{s+2,p}(X, E))$ using Fourier synthesis. By assumption there is an $\epsilon > 0$ such that $\hat{\chi}(k) = 0$ for $|k| < \epsilon$. For $|k| \geq \epsilon$ set

$$\hat{a}_k := (D + |k|^2)^{-1}b.$$

This family depends smoothly on $k \in \mathbf{R}^n$. Define

$$a(x, y) = \int_{\mathbf{R}^n} e^{i\langle x,k\rangle}\hat{a}_k(y)\hat{\chi}(k)\, \mathrm{d}\mathcal{L}^n(k).$$

Then

$$(\Delta_{\mathbf{R}^n} + D)a(x, y) = b\chi.$$

Moreover, one can verify that $x \mapsto \|a(x, \cdot)\|_{W^{2,p}}$ is in $\mathscr{S}(\mathbf{R}^n)$ using a slight variation of the proof that the Fourier transform maps fast decaying functions to fast decaying functions and the estimate $\|\partial_k^\ell \hat{a}_k\|_{W^{s+2,p}} \leq c_\ell(1 + |k|)^\ell \|b\|_{W^{s-2\ell,p}}$.

Step 2. *Let $\chi \in \mathscr{S}(\mathbf{R}^n)$ with $\hat{\chi}(0) = 0$. Then for each $\epsilon > 0$ there is a $\chi_\epsilon \in \mathscr{S}(\mathbf{R}^n)$ which vanishes on $B_\epsilon(0)$ and agrees with χ on $\mathbf{R}^n \backslash B_{2\epsilon}(0)$. The χ_ϵ can be constructed in such a way that $\lim_{\epsilon\to 0} \|\chi_\epsilon - \chi\|_{L^1} = 0$.*

Pick a smooth function $\rho\colon \mathbf{R} \to [0,1]$ such that $\rho(k) = 0$ for $|k| \leq 1$ and $\rho(k) = 1$ for $|k| \geq 2$. Set $\hat{\chi}_\epsilon(k) := \rho(|k|/\epsilon)\chi(k)$ and denote its inverse Fourier transform by χ_ϵ. Then χ_ϵ clearly satisfies the first part of the conclusion. To see that the second part also holds, note that from $\hat{\chi}(0) = 0$ it follows that

$$\|\nabla^{(n)}(\hat{\chi}_\epsilon - \hat{\chi})\|_{L^{2n/(2n-1)}} = O(\epsilon^{\frac{1}{2}})$$

and therefore

$$\begin{aligned}\|\chi_\epsilon - \chi\|_{L^1} &\leq \|(1+|x|)^{-n}\|_{L^{2n/(2n-1)}} \cdot \|(1+|x|)^n(\chi_\epsilon - \chi)\|_{L^{2n}} \\ &\leq c\left(\|\hat{\chi}_\epsilon - \hat{\chi}\|_{L^{2n/(2n-1)}} + \|\nabla^n(\hat{\chi}_\epsilon - \hat{\chi})\|_{L^{2n/(2n-1)}}\right) = O(\epsilon^{\frac{1}{2}})\end{aligned}$$

where $c > 0$ is a constant depending only on n.

Step 3. *Let $1/q = 1 - 1/p$. Then for $\sigma \in \mathscr{S}^n(\mathbf{R}^n)$, $\delta \in \mathbf{R}^n$ and $b \in W^{-s,q}(X, E)$ we have*

$$\int_{\mathbf{R}^n} \langle a(x,\cdot), b\rangle_{W^{s,p},W^{-s,q}} \left(\sigma(x+\delta) - \sigma(x)\right) \,\mathrm{d}\mathcal{L}^n(x) = 0.$$

In particular, the conclusion of the lemma holds.

Let $\chi(x) = \sigma(x+\delta) - \sigma(x)$. Then $\hat{\chi}(0) = 0$. Let χ_ϵ be as in Step 2. Construct c_ϵ such that $(\Delta_{\mathbf{R}^n} + D)c_\epsilon = \chi_\epsilon b$ using Step 1. Then

$$\begin{aligned}\int_{\mathbf{R}^n} \langle a(x,\cdot), b\rangle\, \chi(x) \,\mathrm{d}\mathcal{L}^n(x) &= \lim_{\epsilon\to 0} \int_{\mathbf{R}^n} \langle a(x,\cdot), b\rangle\, \chi_\epsilon(x) \,\mathrm{d}\mathcal{L}^n(x) \\ &= \lim_{\epsilon\to 0} \int_{\mathbf{R}^n}\int_X \langle a(x,y), (\Delta_{\mathbf{R}^n} + D)c_\epsilon\rangle \,\mathrm{d}\mathcal{L}^n(x)\,\mathrm{dvol}(y) \\ &= \lim_{\epsilon\to 0} \int_{\mathbf{R}^n}\int_X \langle (\Delta_{\mathbf{R}^n} + D)a(x,y), c_\epsilon\rangle \,\mathrm{d}\mathcal{L}^n(x)\,\mathrm{dvol}(y) \\ &= 0.\end{aligned}$$

This finishes the proof. □

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